Linear algebra and analytic geometry.

Practice lessons. № 3 The determinants of the second and third order and their calculation. Formula Cramer.

1. Home work: [3], №№1-32, p.123-127 , №№38-50, p.129-130

2. Examples:

2.1. Calculate the determinant of the n-th order:

a) leading to a triangular form.

б) Spreading out on the elements of any row or column on the properties of determinants, we obtain (n-1) zeros in this row or column.

Solution:

a) $$\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 2 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 6 \\ 0 & 5 & 9 & 4 \end{vmatrix} = \frac{1}{4}\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & -1 & 9 \end{vmatrix} \xleftarrow{(-1)} =$$

$$\frac{1}{4}\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & -0 & 2 \end{vmatrix} = \frac{1}{4}\cdot 2\cdot 1\cdot(-1)\cdot 2 = -1$$

б) $$\begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 2 & 1 & 3 & -1 \\ 1 & -1 & 1 & 3 \\ 5 & 1 & 7 & 2 \end{vmatrix} = (-1)(-1)^{1+2}\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 5 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -7 \\ 1 & 0 & 0 \\ 5 & 2 & -13 \end{vmatrix} =$$

$$= (-1)^{2+1}\begin{vmatrix} 1 & 7 \\ 2 & -13 \end{vmatrix} = -(-13+14) = -1$$

2.2. Calculate linear combinations of A and B.

$A = \begin{pmatrix} 2 & 0 & 3 \\ 4 & 7 & 5 \end{pmatrix}$; $B = \begin{pmatrix} 3 & 4 & 1 \\ 8 & 0 & 3 \end{pmatrix}$. Find 5A+2B

Solution:

$$5A + 2B = 5\begin{pmatrix} 2 & 0 & 3 \\ 4 & 7 & 5 \end{pmatrix} + 2\begin{pmatrix} 3 & 4 & 1 \\ 8 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 15 \\ 20 & 35 & 25 \end{pmatrix} + \begin{pmatrix} 6 & 8 & 2 \\ 16 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 16 & 8 & 17 \\ 36 & 35 & 31 \end{pmatrix}$$

2.3 Calculate the AB and BA. Check whether equal to the product of these matrices.

$$A = \begin{pmatrix} 2 & 7 \\ 3 & -1 \\ 0 & 5 \\ 1 & -2 \end{pmatrix}; B = \begin{pmatrix} 7 & 5 & 2 & 1 \\ 4 & -2 & 1 & 0 \end{pmatrix}$$

Solution:

$$AB=\begin{pmatrix}14+28 & 10-14 & 4+7 & 2+0\\ 21-4 & 15+2 & 6-1 & 3+0\\ 0+20 & 0-10 & 0+5 & 0+0\\ 7-8 & 5+4 & 2-2 & 1+0\end{pmatrix}=\begin{pmatrix}42 & -4 & 11 & 2\\ 17 & 17 & 5 & 3\\ -20 & -10 & 5 & 0\\ -1 & 9 & 0 & 1\end{pmatrix}$$

dimension of the matrix AB -(4×4)

$$BA=\begin{pmatrix}14+15+1 & 49-5+10 & -2\\ 8-6+0+0 & 28+2+5 & 0\end{pmatrix}=\begin{pmatrix}30 & 52\\ 2 & 35\end{pmatrix}$$

dimension of the matrix BA - $(2\times 2)\Rightarrow AB\neq BA$

2.4 Find the inverse matrix A^{-1} of matrix A

$$A=\begin{pmatrix}5 & 8 & 3\\ 0 & 2 & 3\\ 0 & 0 & 4\end{pmatrix};$$

Solution: $\det A=40\neq 0$ of matrix A is nondegenerate $\Rightarrow$ can be found A^{-1}

We find $A_{ij}=(-1)^{i+j}M_{ij}$ for a_{ij}.

$$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & 3\\ 0 & 4\end{vmatrix}=8;\ A_{12}=(-1)^{1+2}\begin{vmatrix}0 & 3\\ 0 & 4\end{vmatrix}=0;\ A_{13}=(-1)^{1+3}\begin{vmatrix}0 & 2\\ 0 & 0\end{vmatrix}=0$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}8 & 7\\ 0 & 4\end{vmatrix}=-32;\ A_{22}=(-1)^{2+2}\begin{vmatrix}5 & 4\\ 0 & 4\end{vmatrix}=20,\quad A_{23}=(-1)^{2+3}\begin{vmatrix}5 & 8\\ 0 & 0\end{vmatrix}=0$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}8 & 7\\ 2 & 3\end{vmatrix}=10;\ A_{32}=(-1)^{3+2}\begin{vmatrix}5 & 7\\ 0 & 3\end{vmatrix}=-15;\quad A_{33}=(-1)^{3+3}\begin{vmatrix}5 & 8\\ 0 & 2\end{vmatrix}=10$$

$$A^{T}=\begin{pmatrix}5 & 0 & 0\\ 8 & 2 & 0\\ 7 & 3 & 4\end{pmatrix}$$ transposed matrix

$$\tilde{A}^{T}=\begin{pmatrix}8 & -32 & 10\\ 0 & 20 & -15\\ 0 & 0 & 10\end{pmatrix}$$ adjoint matrix

$$A^{-1}=\frac{1}{\det A}\cdot\tilde{A}^{T}=\frac{1}{40}\begin{pmatrix}8 & -32 & 10\\ 0 & 20 & -15\\ 0 & 0 & 10\end{pmatrix}=\begin{pmatrix}\frac{1}{5} & -\frac{4}{5} & \frac{1}{4}\\ 0 & \frac{1}{2} & -\frac{3}{8}\\ 0 & 0 & \frac{1}{4}\end{pmatrix}$$

check $AA^{-1}=A^{-1}A=E$

$$A\cdot A^{-1}=\begin{pmatrix}5 & 8 & 7\\ 0 & 2 & 3\\ 0 & 0 & 4\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{5} & -\frac{4}{5} & \frac{1}{4}\\ 0 & \frac{1}{2} & -\frac{3}{8}\\ 0 & 0 & \frac{1}{4}\end{pmatrix}=\begin{pmatrix}1+0+0 & -4+4+0 & \frac{4}{5}-3+\frac{7}{4}\\ 0+0+0 & 0+1+0 & 0-\frac{3}{4}+\frac{3}{4}\\ 0+0+0 & 0+0+0 & 0+0+1\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E$$

$$AA^{-1}=\begin{pmatrix}\frac{1}{5} & -\frac{4}{5} & \frac{1}{4}\\ 0 & \frac{1}{2} & -\frac{3}{8}\\ 0 & 0 & \frac{1}{4}\end{pmatrix}\begin{pmatrix}5 & 8 & 7\\ 0 & 2 & 3\\ 0 & 0 & 4\end{pmatrix}=\begin{pmatrix}1 & \frac{5}{8}-\frac{5}{8} & \frac{7}{5}-\frac{12}{5}+1\\ 0 & 1 & \frac{3}{2}-\frac{3}{2}\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=E$$

2.5. Find the rank of matrix

$$A=\begin{pmatrix}4 & 5 & 7 & 2 & 8\\ 1 & 4 & 3 & 8 & 1\\ 2 & 3 & 5 & 7 & 1\end{pmatrix}$$

Solution:

a) Gauss method

$$r(A)=r\begin{pmatrix}1 & 4 & 3 & 8 & 1\\ 2 & 3 & 5 & 7 & 1\\ 4 & 5 & 7 & 2 & 8\end{pmatrix}=r\begin{pmatrix}1 & 4 & 3 & 8 & 1\\ 0 & -5 & -1 & -9 & -1\\ 0 & -11 & -5 & -30 & 4\end{pmatrix}=$$

$$=r\begin{pmatrix}1 & 4 & 3 & 8 & 1\\ 0 & -5 & -1 & -9 & -1\\ 0 & 0 & 14 & -51 & 31\end{pmatrix}=3.\mathit{because}.\det A=\begin{vmatrix}1 & 4 & 3\\ 0 & -5 & -1\\ 0 & 0 & 14\end{vmatrix}=70\neq 0$$

б) by elementary transformations

$$r(A(=\begin{pmatrix}1 & 4 & 3 & 8 & 1\\ 0 & -5 & -1 & -9 & 1\\ 0 & -11 & -5 & -30 & 4\end{pmatrix}=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & -5 & -1 & -9 & -1\\ 0 & -11 & -5 & -30 & 4\end{pmatrix}=$$

$$=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 14 & -5 & 15 & 9\end{pmatrix}=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 14 & 0 & 15 & 9\end{pmatrix}=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1\end{pmatrix}=$$

$$=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}=r\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{pmatrix}=3$$

в) fringing by minors (методом окаймляющих миноров).

$$A=\begin{pmatrix}4 & 5 & 7 & 2 & 8\\ 1 & 4 & 3 & 8 & 1\\ 2 & 3 & 5 & 7 & 1\end{pmatrix}$$

We fix a non-zero minor of the 2-nd order $M_2=\begin{vmatrix}4 & 5\\ 1 & 4\end{vmatrix}=11\neq 0$

Consider fringing Minor of the 3rd order

$$M_3=\begin{vmatrix}4 & 5 & 7\\ 1 & 4 & 3\\ 2 & 3 & 5\end{vmatrix}=80+21+30-56-25-36=14\neq 0$$

you can't make $M_4 \Rightarrow r(A)=3$

Formula Cramer.

Solve the system of equations with the method of Kramer

We consider a system of three equations with three unknowns

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 2 \\ 3x_1 - 3x_2 + 2x_3 = -6 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

Solution:

Here $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $B = \begin{pmatrix} 2 \\ -6 \\ 2 \end{pmatrix}$; $\Rightarrow A \cdot X = B$ matrix form of the system.

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 10 \end{vmatrix} = 10$$

$$\Delta_{x_1} = \begin{vmatrix} 2 & 2 & 3 \\ -6 & -3 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -4 & -1 & 3 \\ -10 & -5 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (-1)^{3+3} \begin{vmatrix} -4 & -1 \\ -10 & -5 \end{vmatrix} = 10.$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -6 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -10 & -4 \\ 0 & 0 & -2 \end{vmatrix} = 20$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -3 & -6 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & -5 & -10 \\ 1 & 0 & 0 \end{vmatrix} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ -5 & -10 \end{vmatrix} -10$$

$$x_1 = \frac{10}{10} = 1; \quad x_2 = \frac{20}{10} = 2; \quad x_3 = \frac{-10}{10} = -1$$

Solution of the problem: $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = -1 \end{cases}$